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## Synchronization of strange nonchaotic attractors

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Strange nonchaotic attractors (SNAs), which are realized in many quasiperiodically driven nonlinear systems, are strange (geometrically fractal) but nonchaotic (the largest nontrivial Lyapunov exponent is negative). Two such identical independent systems can be synchronized by in-phase driving: Because of the negative Lyapunov exponent, the systems converge to a common dynamics, which, because of the strangeness of the underlying attractor, is aperiodic. This feature, which is robust to external noise, can be used for applications such as secure communication. A possible implementation is discussed and its performance is evaluated. The use of SNAs rather than chaotic attractors can offer some advantages in experiments involving synchronization with aperiodic dynamics. [S1063-651X(97)10612-2]

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Pecora and Carroll [1] showed that identical (or nearly identical) nonlinear systems can be made to synchronize if coupled by a common drive signal. If one considers the overall system as separated into drive and response subsystems, then a necessary and sufficient condition for synchronization to occur is that the Lyapunov exponents corresponding to the response subsystem are all negative. This property is robust and is easy to realize in the laboratory [1–3], even when the dynamics of the drive is chaotic and unstable. An application of chaotic synchronization that has been extensively explored is the possibility of secure communications: A number of different schemes based on a variety of coding principles have been proposed [4–7].

The property of synchronization of nonlinear systems is extremely general. One situation where this is most easily achieved is between quasiperiodically driven systems in the regime wherein the dynamics lie on strange nonchaotic attractors (SNAs) [8,9]. The purpose of this Brief Report is to suggest that such systems possess advantages that make them ideal for applications in communications that use aperiodic signals.

SNAs, which are found in quasiperiodically driven systems, are geometrically strange, namely, they are fractal, but the largest nontrivial Lyapunov exponent is negative and hence the dynamics are not chaotic. They can be created through a variety of mechanisms [10] and exist over a range of parameter values (i.e., they are not exceptional or nongeneric). SNAs have been observed in several experimental systems [11,12] and have been verified through the use of power spectral methods and attractor dimension estimates. Although the largest nontrivial Lyapunov exponent is negative, the dynamics are *aperiodic* since the underlying attractor is strange: This makes it difficult to deduce the Lyapunov

exponents, or indeed the *nonchaotic dynamics*, by attractor reconstruction using standard methods.

Synchronization of two such systems is trivial because of the negative Lyapunov exponents. Regardless of where the systems are started, they eventually converge to the same dynamics so long as the *phase* of the quasiperiodic driving is matched. There is no requirement of coupling the systems (other than the coupling implicit in the matched phase; see below).

As an example of this behavior, consider the following system introduced by Zhou, Moss, and Bulsara [13], which describes a driven-damped superconducting quantum interference device:

$$\ddot{x} + k\dot{x} = -(x + \beta \sin 2\pi x) + q_1 \sin \omega_1 t + q_2 \sin \omega_2 t, \quad (1)$$

where the ratio of frequencies is taken to be irrational  $\omega_1/\omega_2 = (\sqrt{5} + 1)/2$ . This system (and related variants) has been extensively studied in both numerical and analog simulations and is thus a typical example of a system that can be experimentally realized. An identical copy of this system with phase difference  $\phi$  has the equation of motion

$$\ddot{y} + k\dot{y} = -(y + \beta \sin 2\pi y) + q_1 \sin \omega_1(t + \phi) + q_2 \sin \omega_2(t + \phi). \quad (2)$$

The two systems can be synchronized regardless of the initial values of  $x, \dot{x}, y, \dot{y}$ , so long as there is no phase lag  $\phi = 0$  and the parameters (here  $k, \beta$  and  $q_1, q_2$ ) are such that the dynamics are on a SNA. Explicitly, it is observed that  $|x(t) - y(t)| \rightarrow 0$  rapidly; results for a typical orbit are shown in Fig. 1(a).

Rewriting the above system in autonomous form

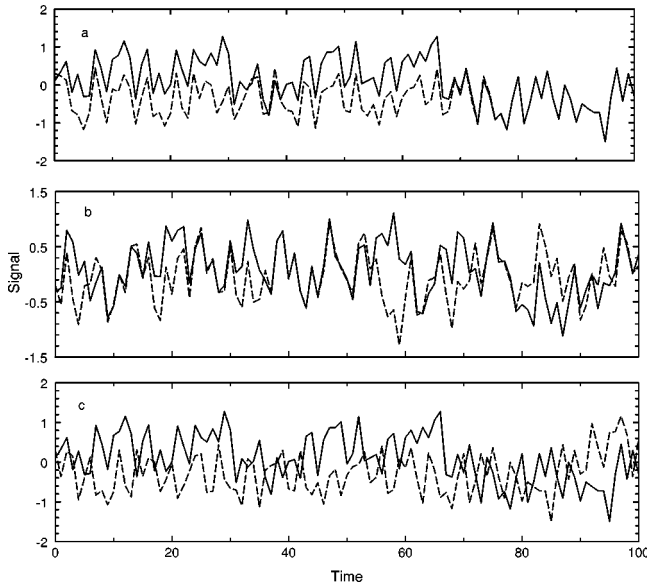


FIG. 1. Time series of the signals from the two systems  $x_1(t)$  (solid line) and  $y_1(t)$  (dashed line). The parameters are set at  $k = \beta = 2$ ,  $q_1 = 2.768$ , and  $\omega_1 = 2.25$ . (a) When  $q_2 = 0.88$  and  $\phi = 0$ , the dynamics are on a SNA and the two systems synchronize. (b) When  $q_2 = 0.38$  and  $\phi = 0$ , the dynamics are on a chaotic attractor. The Lyapunov exponents are all positive and synchronization is not possible. (c) When  $q_2 = 0.88$  and  $\phi \neq 0$ , the dynamics are on a SNA, but synchronization does not occur.

$$\begin{aligned}
 \dot{x}_1 &= x_2, \\
 \dot{x}_2 &= -kx_2 - (x_1 + \beta \sin 2\pi x_1) + q_1 \sin \omega_1 x_3 + q_2 \sin \omega_2 x_3, \\
 \dot{x}_3 &= 1, \\
 \dot{y}_1 &= y_2, \\
 \dot{y}_2 &= -ky_2 - (y_1 + \beta \sin 2\pi y_1) + q_1 \sin \omega_1 y_3 + q_2 \sin \omega_2 y_3, \\
 \dot{y}_3 &= 1
 \end{aligned} \tag{3}$$

makes it evident that phase matching corresponds to replacing  $y_3$  in Eq. (3) by  $x_3$  and thereby coupling the two systems. This then conforms to the general framework of synchronization in the manner of Pecora and Carroll [1] with the  $x$  system the drive and the  $y$  effectively the response.

In other parameter ranges, the system in Eq. (1) can be chaotic. In such a case, both the drive and the response have positive Lyapunov exponents and synchronization cannot occur; see Fig. 1(b). When there is a phase mismatch, namely, if  $\phi \neq 0$  in Eq. (2), again synchronization does not occur [Fig. 1(c)], even when the parameters correspond to SNA dynamics.

Secure communications using aperiodic dynamics has been implemented in several ways [4–7] and the technique of synchronization with SNAs rather than chaotic attractors can be employed in several of them. It should be mentioned, however, that some of the simpler schemes have been shown to be susceptible to unmasking [14] by inference of the underlying attractor.

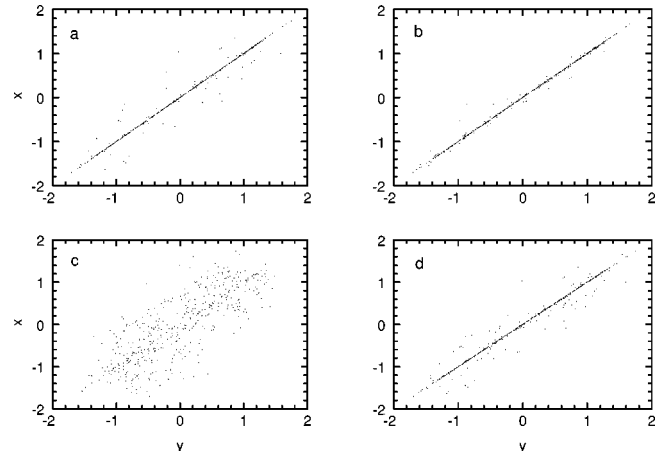


FIG. 2. Robustness of synchronization with respect to noise that alters the parameters of the response system, as described in the text. The values of the parameters are  $k = \beta = 2$ ,  $q_1 = 2.768$ ,  $q_2 = 0.88$ , and  $\omega_1 = 2.25$  and the noise strength is  $\sigma = 10^{-2}$  for (a)  $\mu \equiv \beta$  [see Eq. (4)], (b)  $\mu \equiv q_2$ , and (c)  $\mu \equiv \omega_2$ . Reduction of the noise strength improves the synchronization in the last case (d) where  $\sigma = 10^{-4}$  and  $\mu \equiv \omega_2$ .

The most direct method of secure communication through the use of chaotic synchronization uses a chaotic signal to mask information [4]. The alternate strategy suggested here is to use the signal from a strange nonchaotic system in an analogous manner, by transmitting the low-amplitude information-bearing signal  $m(t)$  that is added to (and masked by) the output from the first system  $x(t)$ , namely,  $x'(t) = x(t) + m(t)$ . It is also necessary to simultaneously transmit a means of phase locking, say, a train of  $\delta$ -function

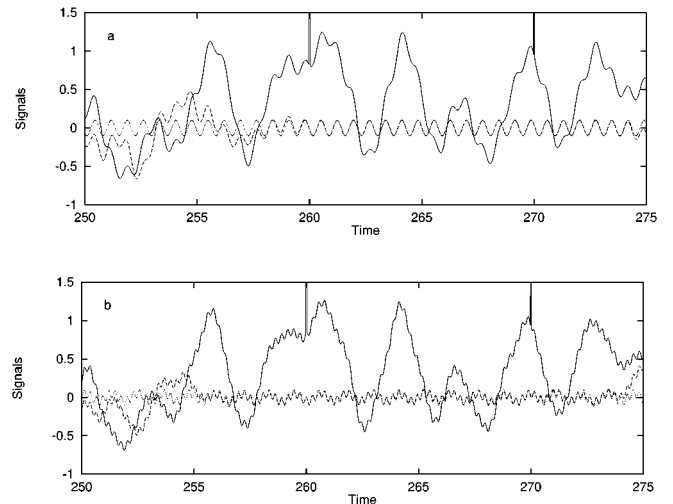


FIG. 3. Demonstration of the viability of the secure communication scheme. (a) The signal being communicated is  $m(t) = 0.1 \sin \omega_2 t$  (dotted line), which is added to the output from the SNA, i.e.,  $x'(t)$  (solid line). The parameter  $q_2$  of the response system differs from that of the drive by 10%. Other SNA parameters are as in Fig. 1(a) and the recovered signal is the dashed curve. (b) The signal being communicated is  $m(t) = 0.1 \sin \omega_2 t \sin \omega_1 t$  (dotted line), which is added to the output from the SNA (solid line). The frequency  $\omega_2$  of the response system has fluctuations, with  $\sigma = 10^{-3}$ . Other SNA parameters are as in Fig. 1(a). The recovered signal is the dashed curve. The  $\delta$ -function spikes in  $x'(t)$  are used by the response system for phase matching.

pulses. The signal  $m(t)$  can be recovered by allowing the systems to synchronize and subtracting the output of the second system, i.e.,  $x'(t) - y(t)$ .

Since the two systems evolve independently, the effect of additive noise is minimal: noise added to  $x'(t)$  will be unchanged upon subtraction. On the other hand, it is necessary to consider the mismatch between the transmitter system and the response in some detail. One way to explore the effect of such mismatch is by introducing fluctuations in the parameters of the response

$$\mu = \mu_0[1 + \sigma\xi(t)], \quad (4)$$

where  $\sigma$  is the noise amplitude,  $\xi(t)$  is a  $\delta$ -correlated random variable with zero mean, and  $\mu_0$  is the value of the parameter in the transmitter system. For the response system in Eq. (2), we consider  $\mu \equiv q_2$ ,  $\beta$ , and  $\omega_2$ . Results are shown in Fig. 2 for the case of noise amplitude  $\sigma = 10^{-2}$  for the three parameters indicated above. The plot of  $x$  vs  $y$  shows that the degree of synchronization in the presence of noise is fairly good, except for the case of  $\mu \equiv \omega_2$ , namely, when the quasiperiodic driving frequency is subject to fluctuations. Indeed, variation of the parameters  $q_2$  and  $\beta$  by up to 10% does not significantly alter the synchronization except for short bursts in time. The drive frequency is much more sensitive to fluctuations and only by reducing the noise amplitude to  $10^{-4}$  is it possible to greatly improve the synchronization in this case [Fig. 2(d)].

The viability of the above scheme is demonstrated using the SNA of Eqs. (1) and (2) and the results are shown in Fig. 3, wherein the signal to be communicated is a sinusoidal form. In the absence of noise, the recovery of the signal is exact (and is therefore not shown); with noise added following Eq. (4) in the driving frequency or in the other parameters, the recovery of the signal is of good quality. Indeed, even when the parameters of the two systems do not match,

signal recovery is possible: In Fig. 3(a) the parameters  $q_2$  of the  $x$  and  $y$  systems differ by 10%. The occasional errors due to loss of synchronization that are apparent in Fig. 3 do not persist over long times.

Note that it is important for this means of application that interception of  $x'(t)$  can have no potential value in the absence of knowledge of the underlying dynamical system since the dynamics is intrinsically aperiodic. One can use standard methods to reconstruct the dynamics [15], but the extraction of reliable values for (small) negative Lyapunov exponents from experimental time-series data for SNAs has proved to be difficult [11,12]. Thus it may be more problematic to reliably reconstruct the underlying attractor, in contrast to the example of the Lorenz system, which was considered by Perez and Cerdeira [14].

Related schemes that use chaotic attractors, for example, the modulation-detection procedure described by Cuomo and Oppenheim [4], can be similarly adapted to the case of SNAs. A somewhat different implementation of secure communications using SNAs that transmits digital information by switching parameter values has also been proposed recently [16].

The synchronizing property arises directly from the use of a common in-phase driving: The negative Lyapunov exponents alone do not guarantee that the  $x$  and  $y$  signals will coincide. (In the extreme case when both systems are integrable, in the absence of a common driving term, there will be no synchronization.) Other applications that use the synchronization of chaotic systems [17] can also be effected using strange nonchaotic systems. In general, as a consequence of the negative Lyapunov exponents, the stability and robustness using SNAs is greater than that with comparable chaotic attractors. This may make quasiperiodically driven systems particularly suitable for applications that involve the synchronization of large numbers of nonlinear dynamical systems.

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